

# CONSTRUCTION OF PENCILS OF EQUIAN- HARMONIC CUBICS

Dissertation

SUBMITTED TO THE BOARD OF UNIVERSITY STUDIES OF THE JOHNS  
HOPKINS UNIVERSITY IN CONFORMITY WITH THE REQUIRE-  
MENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

BY  
JACOB YERUSHALMY

JUNE, 1930

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# CONSTRUCTION OF PENCILS OF EQUIANHARMONIC CUBICS.

By JACOB YERUSHALMY.

1. *Introduction.* By Salmon's theorem, the cross-ratio of the four tangents, which may be drawn to a plane cubic curve from a point on it, is constant for the cubic. A certain function of the cross-ratio, which is rational in the coefficients of the cubic, constitutes the only absolute rational invariant of the cubic. In fact, if  $\alpha$  is one of the values which this cross-ratio assumes, then

$$J = 4(\alpha^2 - \alpha + 1)^3 / (\alpha + 1)^2 (1 - 2\alpha)^2 (2 - \alpha)^2$$

is the absolute rational invariant of the cubic. This invariant is known as the modulus of the cubic.

In terms of the two relative invariants  $S$  and  $T$  of the cubic, which are of degrees 4 and 6 respectively in the coefficients of the cubic,

$$J = S^3 / T^2$$

and hence involves the coefficients to the 12th degree.

If  $\alpha = -1$ , then  $J = \infty$ ,  $T = 0$  and the cubic is said to be *harmonic*.

If  $\alpha = -\epsilon$  ( $\epsilon^3 = 1$ ), then  $J = S = 0$  and the cubic is said to be *equianharmonic*.

If  $\alpha = 1$ , then  $S^3 - T^2 = 0$  and the cubic has a double point.

If  $\alpha$  is indeterminated,  $S = T = 0$  and the cubic has a cusp.

Since  $J$  involves the coefficients of the cubic to the 12th degree, there are in an arbitrary pencil of cubics 12 cubics of an assigned generic modulus, but only six harmonic and four equianharmonic cubics.

2. *Pencils of Cubics of Equal Modulus.* O. Chisini \* proposes to determine all the pencils having the property that all the cubics of one pencil have the same modulus. He succeeds in determining them in the following manner.

He observes that to a pencil of cubics all having the same modulus corresponds a pencil of lines cutting a quartic curve in quadruples of points all having the same cross-ratio, and conversely. He then determines all the

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\* "Sui fasci di cubiche a modulo costante," *Rendiconti del Circolo Matematico di Palermo*, Vol. 41 (1916).

quartic curves admitting the above property and studying their singularities he arrives at the construction of the pencils of cubics.

In the equianharmonic case Chisini proves that every quartic that is cut by all the lines of a pencil in equianharmonic quadruples is represented by an equation of the type

$$f_4 = \frac{\partial^2 \phi_3(x_1 x_2 x_3)}{\partial x_3^2} \phi_3(x_1 x_2 x_3) - \frac{1}{2} \left[ \frac{\partial \phi_3(x_1 x_2 x_3)}{\partial x_3} \right]^2 = 0,$$

where  $\phi_3 = 0$  is an arbitrary cubic not passing thru the center of the pencil of lines which is taken to be the point  $C \equiv (0, 0, 1)$ . It is easily verified that the cubic  $\phi_3 = 0$  is the polar of the point  $C$  with respect to the quartic  $f_4 = 0$ . He also shows that the six points  $C_1, C_2, C_3, C_4, C_5, C_6$  of intersection of  $\phi_3 = 0$  with the polar conic of  $C$  with respect to  $\phi_3 = 0$  are flex-points for  $f_4 = 0$  whose flex-tangents pass thru  $C$ . Obviously the cubic  $\phi_3 = 0$  touches  $f_4 = 0$  at the six points  $C_i$ .

Chisini, however, does not give the actual construction of the pencils of equianharmonic cubics. He only points out that such pencils are characterized by possessing six cuspidal cubics (corresponding to the six flex-tangents  $CC_i$  of the above quartic). It is the object of this paper to make a closer investigation of these pencils of cubics with special reference to their actual construction. We show that every such pencil is contained in a net of equianharmonic cubics thru six base-points. These base-points form the vertices of two in-circumscribed triangles of an arbitrary cubic curve, which are three-fold perspective from the vertices of a third in-circumscribed triangle of the same cubic.

3. *Cremona Transformations Leaving a Pencil of Equianharmonic Cubics Invariant.* It is well known that a general elliptic cubic is transformed into itself by 8 cyclic collineations of period three whose fixed points do not lie on the cubic. These collineations are given in terms of the abelian parameter  $u$  in the form

$$u' = u + \omega/3 \quad (\omega \text{ is a period}).$$

In addition to these general transformations, an equianharmonic cubic is transformed into itself by singular homographies and homologies of period three with three fixed points on the cubic. In terms of the abelian parameter  $u$  these singular transformations are given by

$$u' = + \epsilon u + b, \quad (\epsilon^3 = 1).$$

The existence of such homographies and homologies characterizes the equianharmonic cubics.



If, then, we have a linear system of equianharmonic cubics, say a pencil, the homology may be fixed or variable, i. e., there may exist one homology leaving invariant all the cubics of the pencil, or the homology will vary from cubic to cubic. However Chisini proves (*loc. cit.*, p. 90) that if the homology is fixed the pencil is special, and on a general pencil the homology is variable. It is natural, therefore, to look for Cremona Transformations of the plane into itself leaving the cubics of a pencil of equianharmonic cubics invariant. For this purpose we must investigate more closely the quartic curve  $f_4 = 0$  which is cut by a pencil of lines  $\{C\}$  in equianharmonic quadruples of points.

As is known, a cubic surface  $F_3$  may be mapped on a double plane with a quartic branch-curve by projection from a point  $O$  on the surface. Consider, therefore, the plane  $\pi$  of  $f_4 = 0$  as the projection of a cubic surface  $F_3$ , and  $f_4 = 0$  as the branch-curve. The equation of  $f_4 = 0$  is, as we noted,

$$f_4 = \frac{\partial^2 \phi_3}{\partial x_3^2} \phi_3 - \frac{1}{2} \left( \frac{\partial \phi_3}{\partial x_3} \right)^2 = 0$$

and  $\phi_3 = 0$  is the polar of  $C$  (the center of  $\{C\}$ ) with respect to  $f_4 = 0$ . Take an arbitrary line  $a$  of  $\{C\}$ . It cuts  $f_4 = 0$  in an equianharmonic quadruple of points  $E_1, E_2, E_3, E_4$  and  $\phi_3 = 0$  in three points  $P_1, P_2, P_3$  which constitute the polar group of  $C$  with respect to  $E_1, \dots, E_4$ . To  $a$  will correspond on  $F_3$  a plane equianharmonic cubic curve  $\Psi_3$  cut out by the plane thru  $a$  and  $O$  ( $O$  is the center of projection). To  $P_1, P_2, P_3$  will correspond on  $\Psi_3$  two triples of points. We proceed to prove that each triple of points is a group of three fixed points of a singular birational transformation  $T_3$ , cyclic of order 3, of the cubic  $\Psi_3$  into itself. What we have to prove is, in fact, the following:

**THEOREM.** *The lines joining any point  $O$  on an equianharmonic cubic to the  $\infty^1$  groups of three fixed points, on it, of the  $\infty^1$  singular birational transformations cyclic of period 3, of the cubic into itself form a  $g_3^1$  in the pencil  $\{O\}$ , and this  $g_3^1$  is precisely the  $g_3^1$  obtained by taking the polar group of a variable line of the pencil with respect to the four tangents to the cubic from  $O$ .*

*Proof.* Let  $\Psi_3$  be an equianharmonic cubic and  $O$  a point on it. Let  $\gamma$  be a transformation of  $\Psi_3$  into itself having three fixed points  $A_1', A_2', A_3'$  on the cubic. Let  $\omega$  be the transformation determined by the  $g_2^1$  cut out on  $\Psi_3$  by the pencil  $\{O\}$ , i. e. the transformation which interchanges the points of each pair of the  $g_2^1$ .

Evidently  $\omega^{-1}\gamma\omega$  is a transformation of  $\Psi_3$  into itself having as fixed

points the three further intersections  $A_1'', A_2'', A_3''$  of the lines  $OA_1', OA_2', OA_3'$  with  $\Psi_3$ . Hence given any line of the pencil, each of the two points in which it cuts  $\Psi_3$  outside of  $O$  will define, by a known result, two other points which together with it form a group of three fixed points for some birational transformation of  $\Psi_3$  into itself, and by the previous remark these two triples will form three groups of the  $g_2^1$  mentioned above. It follows that the series  $\infty^1$  of triples of lines mentioned in the theorem is such that each line belongs to one and only one triple, and therefore this series is a  $g_3^1$ .

To prove that this  $g_3^1$  coincides with the  $g_3^1$  of the polar groups we take the trace of  $\{O\}$  on a line  $l$ .

Consider the transformation  $\gamma$  of  $\Psi_3$  into itself having one of its fixed points at the point of contact  $C$  of one of the tangents thru  $O$ .  $\gamma$  will leave invariant the  $g_2^1$  cut out by  $\{O\}$  since it leaves one group of it (the point  $C$  counted twice), invariant, and will permute cyclically the three points of contact of the other 3 tangents to  $\Psi_3$  from  $O$ . The other two fixed points of  $\gamma$  are on a line with  $O$ . We have in  $\{O\}$  a cyclic projectively of order 3. The trace of it on  $l$  may be given by  $x' = \epsilon x$  ( $\epsilon^3 = 1$ ), having the points 0 and  $\infty$  as invariant points. The equianharmonic quadruple of points on  $l$  can be taken to be 0, 1,  $\epsilon$ ,  $\epsilon^2$  and the polar group of any point  $(a_1, a_2)$  with respect to it is given by

$$(1) \quad 4a_1x_1^3 - 3a_2x_1x_2^2 - a_1x_2^3 = 0 \quad \text{or} \quad 4ax^3 - 3x - a = 0.$$

The polar group of the trace of  $OC$  on  $l$  is the polar group of 0 with respect to 0, 1,  $\epsilon$ ,  $\epsilon^2$ . Putting in (1)  $a = a_1/a_2 = 0$  we obtain the point 0 and the point  $\infty$  counted twice, and this triple of points obviously coincides with the traces on  $l$  of the lines joining  $O$  to the three fixed point of  $\gamma$ . The same will hold for any of the 4 groups having one of the 4 points of contact as a fixed point, hence the  $2g_3^1$ 's coincide. q. e. d.

From this theorem we conclude that the two triples of points on  $F_3$  are each a group of fixed points of some birational transformation of  $\Psi_3$  into itself.

To  $\phi_3$  will correspond on  $F_3$  a curve of order 9,  $C_9$ . It will be seen later from the analytical expression for  $F_3$  that  $C_9$  is degenerate, but it can also be seen from the following consideration. Since  $\phi_3 = 0$  touches the branch curve  $f_4 = 0$  whenever they meet, the doubly-covered cubic  $\phi_3 = 0$ , i. e.  $C_9$  does not possess branch-points. Moreover the six points of intersection of  $\phi_3 = 0$  and  $f_4 = 0$  lie on a conic ( $\Psi_2 = \partial\phi_3/\partial x_3 = 0$ ) and by a known theorem \*  $C_9$  is reducible. It will be seen later that  $C_9$  breaks up into a plane

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\* See for instance F. Enriques and O. Chisini, *Courbes et Fonctions Algébriques d'une Variable*, Chap. IV, p. 444.



cubic curve which may be supposed to coincide with the cubic  $\phi_3 = 0$  itself and another sextic curve. The two triples of points on  $F_3$  corresponding to  $P_1, P_2, P_3$  are, hence, separable. One group of 3 points will be on  $\phi_3 = 0$  and the other group will be on the sextic.

Consider only the cubic  $\phi_3 = 0$ . It is traced out by the groups of fixed points of the singular birational transformations, cyclic of order 3, sending into themselves the cubics of the pencil which we obtain on  $F_3$  as the line  $\alpha$  varies in the pencil  $\{C\}$ , i. e., the cubics cut out by the planes thru  $OC$ .

These singular transformations of the cubics of the pencil are, in fact, homologies because the three fixed points on each cubic are on a line (the line of the pencil  $\{C\}$  corresponding to the cubic).

Since thru each point of  $F_3$  there passes only one cubic of the pencil we have a Cremona transformation  $\Gamma$  of the space sending  $F_3$  into itself defined in the following way: For any point  $P$  take the plane of the pencil containing it. In this plane we have an homology sending  $P$  into  $P'$  in the same plane.  $P'$  is the homologous point of  $P$  under  $\Gamma$ . Evidently  $\Gamma$  possesses on  $F_3$  an entire curve of invariant points, the curve  $\phi_3 = 0$  of fixed-points.

4. *The Equation of  $F_3$ .* Consider the equation

$$\frac{\partial^2 \phi_3(x_1 x_2 x_3)}{\partial x_3^2} x_4^2 + (2)^{1/2} \frac{\partial \phi_3(x_1 x_2 x_3)}{\partial x_3} x_4 + \phi_3(x_1 x_2 x_3) = 0.$$

It is an equation of a cubic surface  $F$ . If we project  $F$  from the point  $O = (0, 0, 0, 1)$  on it upon the plane  $x_4 = 0$  we get as branch-curve

$$(\partial^2 \phi_3 / \partial x_3^2) \phi_3 - 1/2 (\partial \phi_3 / \partial x_3)^2 = 0$$

which coincides with our  $f_4 = 0$ .

The following consideration shows that we can really take the above equation to represent our  $F_3$ .

The plane  $x_4 = 0$  is a double plane both for  $F$  and the plane  $\alpha$  of the pencil of equianharmonic cubics (see Chisini, *loc. cit.*, p. 62), the branch-curve being the same in both cases. The net of cubics on  $F$  cut out by the bundle of planes thru  $O$  will be mapped into the net of cubics thru the seven base-points in  $\alpha$ . The system  $\infty^3$  of the plane sections of  $F$  will go by this mapping into the web of cubics thru 6 of the 7 base-points, the seventh base-point will correspond to  $O$ . This web will therefore contain the pencil of equianharmonic cubics and we can take as the equation of  $F_3$ :

$$F_3 \equiv (\partial^2 \phi_3 / \partial x_3^2) x_4^2 + (2)^{1/2} (\partial \phi_3 / \partial x_3) x_4 + \phi_3 = 0.$$

From this equation it is evident that the plane  $x_4 = 0$  cuts out on  $F_3$

the cubic  $\phi_3 = 0$ , and that the  $C_9$  on  $F_3$  corresponding to  $\phi_3 = 0$  breaks up into  $\phi_3 = 0$  itself and another sextic curve.  $\phi_3 = 0$  is, therefore, the curve of invariant points for the Cremona Transformation  $\Gamma$  sending  $F_3$  into itself.

5. *Every Pencil of Equianharmonic Cubics is Contained in a Net of Such Cubics.* The only points on  $F_3$  which may be fundamental points for  $\Gamma$  are the three points  $O, O_1, O_2$ , in which  $OC$  meets  $F_3$ . We proceed to prove, however, that they are ordinary points and consequently  $\Gamma$  possesses no fundamental points on  $F_3$ .

The singular birational transformations sending the cubics of the pencil, cut out on  $F_3$  by the planes on  $OC$ , into themselves are, as we have noted, homologies. Therefore the three fixed points are flexes for the cubic, and it will be shown later that the line joining them is a side of the Hessian triangle of the cubic. The pencil  $\{C\}$  is, hence, composed of the flex lines for the corresponding cubics. Each line forms the axis of the homology; the center being the point common to the three flex tangents at the invariant points.

Since  $O, O_1, O_2$  are not on  $\phi_3 = 0$ , one of the following two cases may happen: either these points go by each homology into different points on the different cubics and will, therefore, be fundamental points for  $\Gamma$ ; or they form for each homology a cycle. This will happen if for each cubic the three flex tangents at the invariant points meet on  $OC$ . In this case  $O, O_1, O_2$  are ordinary points for  $\Gamma$  which possesses, then, no fundamental points on  $F_3$ . To prove that this latter case takes place, it is sufficient, remembering that on a  $P_1, P_2, P_3$  are the polar group of  $C$  with respect to  $E_1, E_2, E_3, E_4$ , to prove the following

**THEOREM.** *The lines joining any point  $O$  of an equianharmonic cubic to any three flexes  $P_1, P_2, P_3$  on a side of the Hessian triangle form the polar group of the line joining  $O$  to the point  $K$  common to the three flex-tangents at  $P_1, P_2, P_3$  with respect to the 4 tangents drawn from  $O$  to the cubic.*

*Proof.* Let the equation of the equianharmonic cubic be

$$\Psi_3 \equiv x_1^3 + x_2^3 + x_3^3 = 0.$$

The Hessian of  $\Psi_3 = 0$  is the cubic  $x_1x_2x_3 = 0$  composed of the three lines  $x_1 = 0, x_2 = 0, x_3 = 0$ .

Let  $O \equiv (y_1, y_2, y_3)$  be any point on  $\Psi_3 = 0$ .

The four tangents from  $O$  to  $\Psi_3 = 0$  are given by

$$3(y_1x_1^2 + y_2x_2^2 + y_3x_3^2)^2 - 4(x_1^3 + x_2^3 + x_3^3)(y_1^2x_1 + y_2^2x_2 + y_3^2x_3) = 0.$$

Let  $P_1, P_2, P_3$  be the three flexes on the line  $x_2 = 0$ . The point  $K$  in this case will be the point  $x_1 = x_2 = 0$  which is the opposite vertex of the flex triangle. Let the pencil of lines  $\{O\}$  be projected upon  $x_2 = 0$ . The three flexes  $P_1, P_2, P_3$  are given by

$$x_1^3 + x_3^3 = 0.$$

The trace of the line  $OK$  is the point  $(y_1, y_3)$ . The equianharmonic quadruple is given by

$$\phi_4 = (y_1x_1^2 + y_3x_3^2)^2 - 4(x_1^3 + x_3^3)(y_1^2x_1 + y_3^2x_3) = 0.$$

The polar triple of  $(y_1, y_3)$  with respect to  $\phi_4 = 0$  is given by

$$y_1\partial\phi_4/\partial x_1 + y_3\partial\phi_4/\partial x_3 = 0$$

or

$$\begin{aligned} & y_1\{12x_1y_1(y_1x_1^2 + y_3x_3^2) - 4y_1^2(x_1^3 + x_3^3) - 12x_1^2(y_1^2x_1 + y_3^2x_3)\} \\ & + y_3\{12x_3y_3(y_1x_1^2 + y_3x_3^2) - 4y_3^2(x_1^3 + x_3^3) - 12x_3^2(y_1^2x_1 + y_3^2x_3)\} = 0 \\ & = 12(y_1x_1^2 + y_3x_3^2)(y_1^2x_1 + y_3^2x_3) - 4(x_1^3 + x_3^3)(y_1^3 + y_3^3) \\ & \quad - 12(y_1x_1^2 + y_3x_3^2)(y_1^2x_1 + y_3^2x_3) = 0 \\ & = x_1^3 + x_3^3 = 0, \end{aligned}$$

which coincides with the three flexes. q. e. d.

We conclude that the three points  $O, O_1, O_2$  form a cycle for each homology, and they are ordinary points for  $\Gamma$ . Moreover, since the point  $K$  in each cubic must form with the three points  $O, O_1, O_2$  an equianharmonic quadruple, it is the same point for all the cubics of the pencil. The transformation  $\Gamma$  has therefore the point  $K$  as an invariant point. Also every point of the plane  $x_4 = 0$  is invariant for  $\Gamma$ , since we have in this plane the pencil of lines  $\{C\}$  each of which is the axis of an homology. We also see that the projectivity involved on the line  $OC$  by the homology on a generic plane on  $OC$  does not vary as the plane varies in the pencil. Hence the transformation  $\Gamma$  does not possess fundamental points, and consequently is a collineation. More precisely,  $\Gamma$  is an homology in space, cyclic of order three, with the point  $K$  as center and  $x_4 = 0$  as plane of invariant points.

The net of cubics cut out on  $F_3$  by the bundle of planes on  $K$  is a net of invariant cubics for  $\Gamma$  since the planes of the bundle are invariant. Each cubic of the net has three invariant points on it (the three points in which it cuts the plane  $x_4 = 0$ ) and is therefore equianharmonic.

By mapping the cubic surface  $F_3$  on the plane  $\alpha$  of the original pencil of equianharmonic cubics we conclude that every pencil of equianharmonic cubics is contained in a net of such cubics thru 6 of the 9 base points of the pencil.



6. *A Canonical Form for  $F_3$  and the Equation of the Homology.* Let us determine the coördinates of the point  $K$ . If we write  $\phi_3 = 0$  in the canonical form

$$\phi_3 \equiv x_1^3 + x_2^3 + x_3^3 + 6\lambda x_1 x_2 x_3 = 0,$$

$F_3$  will take the form

$$F_3 \equiv x_1^3 + x_2^3 + x_3^3 + 6x_3 x_4^2 + 3(2)^{1/2} x_3^2 x_4 + 6(2)^{1/2} \lambda x_1 x_2 x_4 + 6\lambda x_1 x_2 x_3 = 0.$$

Consider the cubic  $\Psi_3$  cut out on  $F_3$  by the plane  $x_1 = \mu x_2$ :

$$\Psi_3 \equiv (1 + \mu^3) x_2^3 + x_3^3 + 6x_3 x_4^2 + 3(2)^{1/2} x_3^2 x_4 + 6(2)^{1/2} \lambda \mu x_2^2 x_4 + 6\lambda \mu x_2^2 x_3 = 0.$$

This is an equianharmonic cubic and its Hessian is composed of a flex triangle. The equation of the Hessian is found to be

$$H \equiv x_4 \{ [(1 + \mu^3) x_2 + 2\lambda \mu (x_3 + (2)^{1/2} x_4)] (x_3 + (2)^{1/2} x_4) - 2\lambda^2 \mu^2 x_2^2 \} = 0.$$

The second factor breaks up into the product of two linear factors by completing the square and taking the difference of the two squares, and we obtain the Hessian triangle. Since  $x_4 = 0$  is one of the sides we conclude that each of the lines of the pencil  $\{C\}$  is a side of the Hessian triangle of the cubic corresponding to it on  $F_3$ .

To determine the coördinates of  $K$ , we consider the cubic cut out on  $F_3$  by the particular plane thru  $x_1 = 0$  and  $O$ . The Hessian triangle of this cubic is found to be

$$x_4 x_2 [(2)^{1/2} x_3 + 2x_4] = 0.$$

The sides of the triangle are the lines

$$x_2 = 0; \quad x_4 = 0; \quad (2)^{1/2} x_3 + 2x_4 = 0.$$

The three tangents at the flexes on  $x_4 = 0$  meet on the opposite vertex, which is the point of intersection of  $x_2 = 0$  and  $(2)^{1/2} x_3 + 2x_4 = 0$ , or in the point  $K \equiv [0, 0, 1, -(2)^{1/2}/2]$ .

For simplicity send the point  $K$  to  $(0, 0, 1, 0)$  and interchange the planes  $x_3 = 0$  and  $x_4 = 0$ .

The transformation sending the points

$$(0, 0, 0, 1), (0, 0, -(2)^{1/2}/2, 1), (0, 0, 1, 0) \text{ into } (0, 0, 0, 1), (0, 0, 1, 0), (0, 0, \epsilon, 1)$$

is found to be  $x_1 = x_1'$ ;  $x_2 = x_2'$ ;  $x_3 = \epsilon x_3'$ ;  $x_4 = (2)^{1/2} (x_4' - \epsilon x_3')$ .

Applying this transformation to  $F_3$  after having interchanged  $x_3$  and  $x_4$  the equation of  $F_3$  turns out to be

$$F_3 \equiv x_1^3 + x_2^3 - 2(2)^{\frac{1}{2}}x_3^3 + 2(2)^{\frac{1}{2}}x_4^3 + 6(2)^{\frac{1}{2}}\lambda x_1x_2x_4 = 0,$$

and the homology in space sending  $F_3$  into itself is evidently

$$\Gamma; \quad x_1' = x_1, \quad x_2' = x_2, \quad x_3' = \epsilon x_3, \quad x_4' = x_4.$$

7. *A Net of Equianharmonic Cubics thru Six Base-Points.* In order to obtain the equianharmonic net of cubics in the plane we map  $F_3$  on a plane  $\alpha$ .

As is known, a cubic surface may be mapped on a plane by means of a system  $\infty^3$  of cubics thru six base-points in the plane, so that the cubics of this system correspond to the plane sections of the cubic surface. The correspondence between the points of the plane and the points of the surface is (1, 1) except for the six base-points  $i$  ( $i = 1, 2, \dots, 6$ ) to which correspond on the surface six lines denoted by  $a_i$ . Also to the six conics thru 5 of the base-points correspond on the cubic surface six lines  $b_i$  (the line  $b_i$  corresponding to the conic leaving out the point  $i$ ). Finally to the 15 lines joining two of the base points  $i$  and  $j$  correspond on the surface the lines  $c_{ij}$ . In all we have 27 lines on the cubic surface.

$F_3$  is a particular cubic surface since there are collineations sending it into itself, and when mapped on  $\alpha$  the six base points will be in particular position which we proceed to determine.

Evidently to  $\Gamma$  will correspond in  $\alpha$  a transformation  $T$ . For a point  $P$  on  $F_3$  is mapped into a point  $Q$  in  $\alpha$ .  $\Gamma$  sends  $P$  into a point  $P'$  which is mapped into a point  $Q'$  in  $\alpha$ . The correspondence between  $Q$  and  $Q'$  is (1, 1) and algebraic, and is therefore a Cremona transformation in the plane.

Since  $\Gamma$  has no fundamental points on  $F_3$ , the Cremona transformation  $T$  will have no fundamental points in  $\alpha$  outside the six base-points. Let us determine the order of  $T$ , i. e., the order of the curves into which  $T$  transforms the lines of  $\alpha$ .

Consider an arbitrary line  $a$  of  $\alpha$ . It is mapped into a twisted cubic  $C$  on  $F_3$ .  $\Gamma$  sends  $C$  into another twisted cubic  $\bar{C}$  which is mapped into a curve  $S$  of  $\alpha$ . As is known there are seventy two systems  $\infty^2$  of twisted cubics on  $F_3$ . One system is mapped into the lines of  $\alpha$ . Twenty systems are mapped into conics thru three of the base-points, thirty systems into cubics with a double point at one of the base-points and passing simply thru four of the remaining five base-points. Twenty systems go into quartics with double points in three of the base-points and simple points at the other three base-points, and finally one system is mapped into quintics having double-points at the six base-points. To determine into what system  $\Gamma$  transforms the system of cubics corresponding to the lines of  $\alpha$  we must find the number of intersections of  $C$  and  $\bar{C}$ .

$C$  cuts  $x_3 = 0$  in three points which are invariant for  $\Gamma$  and belong, hence, also to  $\bar{C}$ . If  $C$  and  $\bar{C}$  have any more common points they must be such points on  $C$  which are the transform of points on  $\bar{C}$ . A point and its transform by  $\Gamma$  must be on a line with  $K$ . However, there is only one double-secant of  $C$  thru  $K$ , and since  $\Gamma$  is cyclic of order three, one of the two points in which the double-secant cuts  $C$  must go by it into the other point. Hence, one of these two points is common to  $C$  and  $\bar{C}$ . These two curves intersect therefore in four points, and so do  $S$  and  $\alpha$  in  $\alpha$ . Hence  $S$  is a quartic curve passing simply thru three of the base-points, say 1, 2, 3 and doubly thru the other base-points 4, 5, 6. The transformation is biquadratic and the fundamental points of the inverse transformation ( $T^{-1}$ ) coincide with those of the direct transformation. The homoloidal net  $\Sigma'$  of the inverse transformation being known

$$\Sigma': (1^1, 2^1, 3^1, 4^2, 5^2, 6^2)_4,$$

it is easy to determine the homoloidal net  $\Sigma$  of  $T$ .  $T$  being cyclic of order three must transform the system  $\Sigma'$  into  $\Sigma$ . A quartic  $C_4$  of  $\Sigma$  must therefore have four variable intersections with a quartic  $C_4'$  of  $\Sigma'$ . Let  $\Sigma$  have multiplicities  $r_1, r_2, r_3$  at 1, 2, 3 and  $s_1, s_2, s_3$  at 4, 5, 6. We have

$$r_1 + r_2 + r_3 + 2(s_1 + s_2 + s_3) = 16 - 4 = 12$$

and since three of them must have the value two and the other three must have the value one, we have the result

$$r_1 = r_2 = r_3 = 2, \quad s_1 = s_2 = s_3 = 1,$$

and hence

$$\Sigma: (1^2, 2^2, 3^2, 4^1, 5^1, 6^1)_4.$$

Corresponding to the cubic  $\phi_3 = 0$  on  $F_3$ , we have in the system  $\infty^3$  of cubics in  $\alpha$  a cubic  $\phi$  each point of which is invariant for  $T$ . Consider the base-point 4. It is a simple fundamental point and goes by  $T$  into a line  $l$  which has no variable intersections with the quartics of  $\Sigma'$  and is therefore on two of the three points 4, 5, 6. This says that on  $F_3$ ,  $a_4$  goes into  $c_{ij}$  ( $ij = 4, 5, 6$ ) by  $\Gamma$ . But  $c_{ij}$  must pass thru the point of intersection of  $a_4$  and  $\phi_3 = 0$  and therefore the line  $l$  in  $\alpha$  passes thru 4. It may be either  $(45)_1$  or  $(46)_1$ . Let  $T(4) = (45)_1$ , then  $T(5) = (56)_1$ ,  $T(6) = (64)_1$ . Moreover, since  $\phi_3 = 0$  and  $c_{45}$  have a point in common on  $a_4$ , the line  $(45)_1$  and the cubic  $\phi$  have at 4 the same direction.  $(45)_1$  is, hence, tangent to  $\phi$  at 4. In the same way it is seen that  $(56)_1$  and  $(64)_1$  are tangent to  $\phi$  at 5 and 6 respectively. The three points 4, 5, 6 form therefore an in-circum-



scribed triangle in the cubic  $\phi$ . Applying the same reasoning for  $T^{-1}$  we see that the other three fundamental points 1, 2, 3 form the vertices of another in-circumscribed triangle in the same cubic.

The point 1 goes by  $T$  into a conic passing thru 1. This conic goes thru 4, 5, 6 since it has no variable intersections with the quartics of  $\Sigma'$ . 1 is transformed, hence, into either  $(12456)_2$  or  $(13456)_2$ . Let  $T(1)=(12456)_2$  then  $T(2)=(23456)_2$  and  $T(3)=(31456)_2$ . We have for  $T$

$$\begin{aligned} 1 &\rightarrow (12456)_2 \rightarrow (13)_1 \rightarrow 1 \\ 2 &\rightarrow (23456)_2 \rightarrow (12)_1 \rightarrow 2 \\ 3 &\rightarrow (31456)_2 \rightarrow (23)_1 \rightarrow 3 \\ 4 &\rightarrow (45)_1 \rightarrow (12346)_2 \rightarrow 4 \\ 5 &\rightarrow (56)_1 \rightarrow (12345)_2 \rightarrow 5 \\ 6 &\rightarrow (64)_1 \rightarrow (12356)_2 \rightarrow 6. \end{aligned}$$

In order to obtain the complete configuration of the six base-points, consider for example the point 1. We have by  $T$

$$1 \rightarrow (12456)_2 \rightarrow (13)_1 \rightarrow 1.$$

On  $F_3$  we have the cycle  $(a_1 b_3 c_{31})$ . These three lines must concur at a point on  $\phi_3 = 0$  (the invariant point on  $a_1$ ), and hence both the conic  $(12456)$  and the line  $(31)$  are tangent to  $\phi$  at 1. The same is true for the other five conics. The complete configuration of the six base-points is therefore as follows:  $(1, 2, 3)$   $(4, 5, 6)$  are the vertices of two in-circumscribed triangles on  $\phi$ , such that the lines  $(13)$ ,  $(12)$ ,  $(23)$ ,  $(45)$ ,  $(56)$ ,  $(64)$  and the conics  $(12456)$ ,  $(23456)$ ,  $(31456)$ ,  $(12346)$ ,  $(12345)$ ,  $(12356)$  are tangent to  $\phi$  at the points 1, 2, 3, 4, 5, 6 respectively.

As is known there are twenty-four in-circumscribed triangles in an arbitrary cubic  $\phi$ . For, if  $(4, 5, 6)$  be such a triangle, and  $u$  the abelian parameter of the cubic chosen so that for three points on a line the sum of the values of this parameter should be equal to zero (mod. periods), then we have

$$\begin{aligned} 2u_4 + u_5 &\equiv 0 \pmod{\omega, \omega'} \\ 2u_5 + u_6 &\equiv 0 \pmod{\omega, \omega'} \\ 2u_6 + u_4 &\equiv 0 \pmod{\omega, \omega'}. \end{aligned}$$

From which

$$u_4 \equiv (\alpha\omega + \alpha'\omega')/9.$$

Giving to  $\alpha$  and  $\alpha'$  all the values from 0 to 8 we get 81 points from which we have to exclude the 9 flexes, leaving 72 points forming 24 triangles.

If we start with any ninth of a period for  $u_4$ , then  $u_5$  and  $u_6$ , and hence

the triangle (4, 5, 6), are determined. In order to determine the second triangle (1, 2, 3) we must put the condition that  $(12346)_2$  should touch  $\phi$  at 4. Hence

$$(2) \quad 2u_4 + u_5 + u_1 + u_2 + u_3 \equiv 0 \pmod{\omega, \omega'}.$$

But since (1, 2, 3) is also an in-circumscribed triangle, the following relations exist

$$\begin{aligned} u_3 &\equiv -2u_1 \pmod{\omega, \omega'} \\ u_2 &\equiv 4u_1 \pmod{\omega, \omega'}. \end{aligned}$$

Substituting in (2) we have

$$\begin{aligned} 2u_4 + u_1 &\equiv (\beta_\omega + \beta'_{\omega'})/3, \\ u_1 &\equiv -2u_4 + \frac{1}{3} \text{ of a period} \equiv u_4 + \frac{1}{3} \text{ of a period.} \end{aligned}$$

We have 9 thirds of a period, and it would seem that associated with any triangle there are 8 more each of which together with it will give the required six points. However, finding the values of  $u_2$  and  $u_3$

$$\begin{aligned} u_2 &\equiv 4u_4 + \frac{1}{3} \text{ of a period} \equiv u_4 + \frac{1}{3} \text{ of a period} \\ u_3 &\equiv -8u_4 + \frac{1}{3} \text{ of a period} \equiv u_4 + \frac{1}{3} \text{ of a period} \end{aligned}$$

we see that these also are obtained by adding to  $u_4$  a third of a period, and hence the nine thirds of a period give rise to only 3 triangles. One triangle is to be excluded; this is the one obtained by adding to  $u_4$  the third periods  $0, 3u_4, 6u_4$ , which give the triangle (4, 5, 6) over again. Hence, starting with any one point (a ninth of a period), the triangle containing it is determined and with it there are determined two associated triangles, each of which can be taken with the original one to form the base of the net.

In fact, start with a ninth of a period  $u_1 \equiv \omega/9$ . The three triangles are

$$\begin{aligned} u_1, & \quad u_2 \equiv u_1 + \omega/3, & \quad u_3 \equiv u_1 + 2\omega/3; \\ u_4 \equiv u_1 + \omega'/3, & \quad u_5 \equiv u_1 + 2\omega/3 + \omega'/3, & \quad u_6 \equiv u_1 + \omega/3 + \omega'/3; \\ u_7 \equiv u_1 + 2\omega'/3, & \quad u_8 \equiv u_1 + 2\omega/3 + 2\omega'/3, & \quad u_9 \equiv u_1 + \omega/3 + 2\omega'/3. \end{aligned}$$

It is easily verified from the above that any two of the three triangles satisfy our conditions, and any two of them can be taken to form the base.

It is also easily seen that any two of the above three triangles are three-fold perspective from the vertices of the third. For

$$\begin{aligned} u_1 + u_4 + u_8 &\equiv u_2 + u_5 + u_3 \equiv u_3 + u_6 + u_8 \equiv 0; \\ u_1 + u_5 + u_7 &\equiv u_2 + u_6 + u_7 \equiv u_3 + u_4 + u_7 \equiv 0; \\ u_1 + u_6 + u_9 &\equiv u_2 + u_4 + u_9 \equiv u_3 + u_5 + u_9 \equiv 0. \end{aligned}$$

Finally we want to show that if we take on an arbitrary cubic  $\phi$  six points forming the vertices of two in-circumscribed triangles and satisfying the above conditions, then in the system  $\infty^3$  of cubics thru these points there is contained a net of equianharmonic cubics.

Consider a biquadratic transformation  $T$  having these six points as fundamental points  $(1^2, 2^2, 3^2, 4^1, 5^1, 6^1)$ . The fundamental points of the inverse transformation will generally be some other six points  $(1_0^1, 2_0^1, 3_0^1, 4_0^2, 5_0^2, 6_0^2)$  of the plane.  $T$  sends the cubics thru  $(1, 2, \dots 6)$  into the cubics thru  $(1_0, 2_0, \dots 6_0)$ . It will send  $\phi$  into some cubic  $\phi_0$ .

The quartics of  $\Sigma'$  having double-points at 1, 2, 3 and passing simply thru 4, 5, 6 cut the cubic  $\phi$  in  $\infty^2$  groups of 3 points. But since

$$\begin{aligned} 2(u_1 + u_2 + u_3) &\equiv 2\omega/3, \\ u_4 + u_5 + u_6 &\equiv \omega/3, \end{aligned}$$

and hence

$$2(u_1 + u_2 + u_3) + u_4 + u_5 + u_6 \equiv 0,$$

it follows that the sum of the Abelian parameters at each group of three points is equal to zero, i. e. the three points of each group are on a line.  $\Sigma'$  cuts, therefore, out on the cubic  $\phi$  the same  $g_3^2$  cut out on it by the lines of the plane. The transformation  $T$  involves, therefore, a collineation  $\gamma$  sending  $\phi$  into  $\phi_0$ .  $\gamma$  sends the points 1, 2,  $\dots$  6 into the points  $1_0, 2_0, \dots 6_0$ . To prove this take for example the line (13).  $T$  sends it into one of the points  $1_0, 2_0, 3_0$  say into  $1_0$ . Consider the pencil of quartics in  $\Sigma'$  degenerating into the fixed line (13) and the pencil of cubics  $(2^2, 4^1, 5^1, 6^1, 1^1, 3^1)$ . Then  $g_3^1$  cut out by it on  $\phi$  has a fixed point which falls at 1 since (13) is tangent to the cubic  $\phi$  at this point. To it will correspond the  $g_3^1$  cut out on  $\phi_0$  by the lines on  $1_0$ , and the fixed point  $1_0$  corresponds to the fixed point 1. And by the same reasoning  $\gamma$  send 2 into  $2_0$  etc.

Multiplying  $T$  by  $\gamma^{-1}$  we get a Cremona transformation  $R$  having the fundamental points of both the direct and inverse transformations coincident. This transformation will leave invariant the linear system  $\infty^3$  of cubics on 1,  $\dots$  6. Moreover, each point of the cubic  $\phi$  is an invariant point for  $R$ .

The cube of this transformation is a collineation in the plane. Because the lines of the plane go by it into the quartics of  $\Sigma'$ :  $(1^1, 2^1, 3^1, 4^2, 5^2, 6^2)$ , which go by the same transformation into quartics having four variable intersections with those of  $\Sigma'$ .  $R$  sends, therefore,  $\Sigma'$  into  $\Sigma$ . Applying  $R$  to  $\Sigma$  we get again the lines of the plane. But a collineation having a cubic of invariant points is the identity, hence  $R^3 = I$  and the transformation is cyclic of period 3.



If we consider then the linear space  $S_3$  whose elements are the cubics of our system  $\infty^3$ , our transformation  $R$  will be a cyclic projectivity because it is an algebraic  $(1, 1)$  correspondence between the elements. This projectivity has one invariant point (the cubic  $\phi$ ) and also every line on this point is invariant: in fact every pencil of cubics obtained by combining linearly any cubic of the system with  $\phi$  goes into itself since the three base-points of the pencil outside the fundamental points are invariant. By duality, since the projectivity has a bundle of invariant lines, it must have a plane of invariant points. The transformation  $R$  in the plane has, therefore, a net of invariant cubics. Each of the cubics of this net has three fixed points on it (the three points in which it cuts the cubic  $\phi$ ) and is therefore equianharmonic.

It is of interest to note that the cubic  $\phi$  is the locus of the cusps of all the cuspidal cubics of the net, because the cusp must be an invariant element and is therefore on the invariant cubic.

### VITA.

Jacob Yerushalmy was born on August 5, 1904, at Vienna, Austria. He received his elementary education in the school of Gederah, Palestine and his high school work was done at the Hebrew College ("Gymnasia Ivrit") Jerusalem, Palestine. He spent the year 1925-26 in the Institute of Polytechnics, Brooklyn, N. Y. In October, 1926, he entered the collegiate department of the Johns Hopkins University from which he received his A. B. degree in 1927. Since that time he has been a student in the graduate department of Mathematics in the University receiving his M. A. degree in 1929. He was Junior Instructor in Mathematics since February, 1929.

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